

On the absolute irreducibility of hyperplane sections of generalized Fermat varieties in \mathbb{P}^3 and the conjecture on exceptional APN functions: the Kasami-Welch degree case

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Abstract

Let f be a function on a finite field F . The decomposition of the generalized Fermat variety X defined by the multivariate polynomial of degree n , $\phi(x, y, z) = f(x) + f(y) + f(z)$ in $\mathbb{P}^3(\overline{\mathbb{F}}_2)$, plays a crucial role in the study of almost perfect non-linear (APN) functions and exceptional APN functions. Their structure depends fundamentally on the Fermat varieties corresponding to the monomial functions of exceptional degrees $n = 2^k + 1$ and $n = 2^{2k} - 2^k + 1$ (Gold and Kasami-Welch numbers, respectively). Very important results for these have been obtained by Janwa, McGuire and Wilson in [12, 13]. In this paper we study X related to the Kasami-Welch degree monomials and its decomposition into absolutely irreducible components. We show that, in this decomposition, the components intersect transversally at a singular point.

This structural fact implies that the corresponding generalized Fermat hypersurfaces, related to Kasami-Welch degree polynomial families, are absolutely irreducible. In particular, we prove that if $f(x) = x^{2^{2k} - 2^k + 1} + h(x)$, where $\deg(h) \equiv 3 \pmod{4}$, then the corresponding

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APN multivariate hypersurface is absolutely irreducible, and hence $f(x)$ is not exceptional APN function. We also prove conditional result in the case when $\deg(h) \equiv 5 \pmod{8}$. Since for odd degree $f(x)$, the conjecture needs to be resolved only for the Gold degree and the Kasami-Welch degree cases our results contribute substantially to the proof of the conjecture on exceptional APN functions—in the hardest case: the Kasami-Welch degree.

Keywords: almost perfect nonlinear (APN), Exceptional APN function conjecture, cyclic codes, Deligne estimate, Lang-Weil estimate, Ghorpade-Lachaud estimate, absolutely irreducible polynomial, Fermat variety, CCZ-equivalence, EA-equivalence, Gold function, Kasami function

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1 Introduction

An almost perfect nonlinear (APN) function (necessarily a polynomial function) on a finite field \mathbb{F} is a non-linear function that is very useful in cryptography because of its excellent resistance to differential cryptanalysis as was demonstrated by Nyberg and Knudsen [17]. APN functions can be related to a host of problems in coding theory, sequence design, exponential sums, projective geometry, block designs, and permutation polynomials. Therefore APN functions were well studied even before Nyberg defined them, and by now are well known objects of research. In this article, we make substantial progress towards the resolution of the main conjecture (stated at the end of this section) on the existence of exceptional APN functions on finite fields. We also contribute to the understanding of the mysteries surrounding this conjecture.

Up until now, the main tool used by most researchers in the study of exceptional APN functions, has been the method of Janwa, McGuire and Wilson in [12] to prove the absolute irreducibility of multivariate polynomials. The algorithmic approach in [12] is based on intersection multiplicity theory and Bezout's theorem, and computations initiated by Janwa and Wilson in [13].

Delgado and Janwa, in [4, 5, 6], based their techniques of proving absolute irreducibility on repeated hyperplane intersections, linear transformations, reductions, and properties of known APN monomial functions. These allowed

us to overcome the very difficult multiplicity computations for hypersurfaces in the projective space \mathbb{P}^3 . As a consequence, we established absolute irreducibility of a class of Gold degree multivariate polynomials over finite fields. In this article, our methods for proving absolute irreducibility are based on the decompositions of symmetric varieties as well as the transversal intersections of its components. This methods permit us to prove the absolute irreducibility of new class of Kasami-Welch degree multivariate polynomials, which implies a contribution on perhaps the hardest case of the Exceptional APN conjecture, as stated in the abstract

The first step in many applications of algebraic geometry to coding theory, cryptography, number theory and other disciplines, is the demonstration of absolute irreducibility of a given variety. Indeed, absolute irreducibility is a necessary condition in the applications of the bounds of Weil, Bombieri, Deligne, Lang-Weil, Ghorpade-Lachaud, and others that estimate the number of rational points on the corresponding varieties, or give bounds on exponential sums along curves. Except for the well known Eisenstein criterion (applicable in very restrictive cases), only few scattered results were known for proving absolute irreducibility (see Schmidt [21]—mostly applicable to Kummer and Artin-Schreier type of extensions), before the major breakthrough in [12].

Therefore, our techniques (in this paper and in [4, 5, 6, 7]) and results are of independent interest. In essence, our techniques are precursor to what would be multiplicity analysis in higher dimensions. These techniques could be used to prove absolute irreducibility of other multivariate polynomials (for possible applications, see Lidl and Niederreiter [16]).

An important fact that we should remark is that our results are far better than other previous results on Gold and Kasami-Welch degree multivariate polynomials. The results of Aubry, McGuire, Rodier, Ferard and Oyono, in [1, 19, 10], provides absolutely irreducible families of polynomials with a big gap between the two higher degree terms, ruling out a considerable number of members of the families. Delgado and Janwa in [6] and in this article (see Theorems 5 and 6) overcame this obstacle and establish absolute irreducibility for almost all Gold and Kasami-Welch degree polynomials of the form $x^n + h(x)$ where degree of h is an odd number. Thus, our results contribute significantly to the proof of the conjecture.

Some of the generalized Fermat hypersurfaces are quite interesting. The monomial hypersurfaces correspond to cyclic codes. For example, the monomial x^7 leads to the Klein Quartic, whose zeta function has been computed,

and applied to determine the number of codewords of weight four in the corresponding cyclic codes in [14]. Similarly, the other hypersurfaces can be used to analyze weight distribution of the corresponding codes. Some of the absolutely irreducible multivariate polynomials could also be used for the construction of algebraic geometric codes from the corresponding curves and surfaces.

Definition 1. [17] Let $L = \mathbb{F}_q$, with $q = p^n$ for positive integer n . A function $f : L \rightarrow L$ is said to be **almost perfect nonlinear** (APN) on L if for all $a, b \in L$, $a \neq 0$, the following equation

$$f(x + a) - f(x) = b \quad (1)$$

has at most 2 solutions.

Equivalently, for $p = 2$, f is APN if the cardinality of the set $\{f(x + a) - f(x) : x \in L\}$ is at least 2^{n-1} for each $a \in L^*$. The best known examples of APN functions are the Gold function $f(x) = x^{2^k+1}$, and the Kasami-Welch function $f(x) = x^{2^{2k}-2^k+1}$; they are APN on any field \mathbb{F}_{2^n} when k and n are relatively prime. The Welch function $f(x) = x^{2^r+3}$ is also APN on \mathbb{F}_{2^n} when $n = 2r + 1$ (see [13] and [12]).

The APN property is invariant under transformations of functions.

A function $f : L \rightarrow L$ is linear if and only if f is a linearized polynomial over L , that is,

$$f(x) = \sum_{i=0}^{n-1} c_i x^{p^i}, \quad c_i \in L.$$

The sum of a linear function and a constant is called an affine function.

Two functions f and g are called *extended affine equivalent* (EA-equivalent), if $f = A_1 \circ g \circ A_2 + A$, where A_1 and A_2 are linear maps and A is a constant function. They are called CCZ-equivalence, if the graph of f can be obtained from the graph of g by an affine permutation. EA-equivalence is a particular case of CCZ-equivalence; two CCZ-equivalent functions preserve the APN property (for more details see [2]). In general, CCZ-equivalence is very difficult to establish.

Until 2006, the list of known affine inequivalent APN functions on $L = \mathbb{F}_{2^n}$ was rather short; the list consisted only of monomial functions of the form $f(x) = x^t$, for positive integer t . In February 2006, Y. Edel, G. Kyureghyan and A. Pott [8] established (by an exhaustive search) the first example of an

APN function not equivalent to any of the known monomial APN functions. Their example is

$$x^3 + ux^{36} \in \mathbb{F}_{2^{10}}[x],$$

where $u \in w\mathbb{F}_{2^5}^* \cup w^2\mathbb{F}_{2^5}^*$ and w has order 3. It is APN on $\mathbb{F}_{2^{10}}$. Since then several APN polynomials have shown to be APN and not CCZ-equivalent to known power functions (by Felke, Leander, Bracken, Budaghyan, Byrne, Markin, McGuire, Dillon and others). This example, and other results opened up the possibility that there are perhaps other sequences of APN function than the monomial APN functions that have Gold and Kasami exponents.

Definition 2. Let $L = \mathbb{F}_q$, with $q = p^n$ for positive integer n . A function $f : L \rightarrow L$ is called **exceptional APN** if f is APN on L and also on infinitely many extensions of L .

From now on, we will assume $p = 2$. The main conjecture on APN functions, formulated by Aubry, McGuire and Rodier [1] is the following:

CONJECTURE: Up to equivalence, the Gold and Kasami-Welch functions are the only exceptional APN functions.

Rodier provided a characterization for APN functions using varieties (see [18]).

Let $L = \mathbb{F}_q$, with $q = 2^n$. A function $f : L \rightarrow L$ is APN if and only if the affine variety X with equation

$$f(x) + f(y) + f(z) + f(x + y + z) = 0$$

has all its rational points contained in the surface $(x + y)(x + z)(y + z) = 0$.

Using this characterization, Rodier provided the following criteria about exceptional APN functions. (The bound results of Lang-Weil and Ghorpade-Lachaud about rational points on a surface are crucial for the proof of the next Theorem)

Theorem 1. *Let $f : L \rightarrow L$, $L = \mathbb{F}_{2^n}$, a polynomial function of degree d . Suppose that the variety X of affine equation*

$$\frac{f(x) + f(y) + f(z) + f(x + y + z)}{(x + y)(x + z)(y + z)} = 0$$

is absolutely irreducible (or has an absolutely irreducible component over L), then f is not an exceptional APN function.

As can be seen in this theorem, proving absolute irreducibility, or the existence of an absolutely irreducible factor, does guarantee not to be exceptional APN. In section 3 we study X related to the Kasami-Welch number (the Kasami-Welch variety X), its decomposition into absolutely irreducible components, as well as some properties of these components. One of our main results is that these components intersect transversally at a particular point. This result provides two new infinite families of absolutely irreducible polynomials in section 4 and, in section 5, as a direct application, its contribution to the conjecture of exceptional functions.

For the rest of the article, let us denote

$$\phi(x, y, z) = \frac{f(x) + f(y) + f(z) + f(x + y + z)}{(x + y)(x + z)(y + z)}, \quad (2)$$

$$\phi_j(x, y, z) = \frac{x^j + y^j + z^j + (x + y + z)^j}{(x + y)(x + z)(y + z)} \quad (3)$$

Thus, if $f(x) = x^d + a_{d-1}x^{d-1} + a_{d-2}x^{d-2} + \dots + a_0$, then

$$\phi(x, y, z) = \sum_{j=3}^d a_j \phi_j(x, y, z) \quad (4)$$

Let us call $\phi(x, y) = \phi(x, y, 1)$, $\phi_j(x, y) = \phi_j(x, y, 1)$, its affine parts.

2 The Kasami-Welch case of the conjecture of APN functions

The well known conjecture [1] about exceptional APN functions is:

CONJECTURE: Up to equivalence, the Gold and Kasami-Welch monomial functions, $f(x) = x^{2^k+1}$ and $f(x) = x^{2^{2k}-2^k+1}$ respectively, are the only exceptional APN functions.

The conjecture was settled for monomials by Hernando and McGuire in [11]. Important results supporting this conjecture have been obtained in the last years for polynomials. In [1], Aubry, McGuire and Rodier settled the conjecture for odd degree polynomials (excluding the Gold and Kasami-Welch degree). Very recently, many important results has been also obtained for Gold degree polynomials by Delgado and Janwa in [4, 5, 6]. The Kasami-Welch case has been hardly studied up to now. Ferard, Oyono and Rodier in [10] obtained the following results (the only two established for this case).

Theorem 2. Suppose that $f(x) = x^{2^{2k}-2^k+1} + g(x) \in L[x]$ where $\deg(g) \leq 2^{2k-1} - 2^{k-1} + 1$. Let $g(x) = \sum_{j=0}^{2^{2k-1}-2^{k-1}+1} a_j x^j$. Suppose moreover that there exist a nonzero coefficient a_j of $g(x)$ such that $\phi_j(x, y, z)$ is absolutely irreducible. Then f is not exceptional APN.

For the case when the degree of g is $2^{2k-1} - 2^{k-1} + 2$, they obtained:

Theorem 3. Suppose that $f(x) = x^{2^{2k}-2^k+1} + g(x) \in L[x]$ where $\deg(g) \leq 2^{2k-1} - 2^{k-1} + 2$. Let $k \geq 3$ be odd and relatively prime to n . If $g(x)$ does not have the form $ax^{2^{2k-1}-2^{k-1}+2} + a^2x^3$ then ϕ is absolutely irreducible, while if $g(x)$ does have this form then either ϕ is irreducible or ϕ splits into two absolutely irreducible factors which are both defined over L .

Unlike the Gold case, for the Kasami case, it has been difficult to push up on the degree of $g(x)$ to obtain new results. Part of this difficulty is due to the scarce knowledge on the decomposition of the Kasami-Welch variety.

3 Transversal intersection on the Kasami-Welch variety

Let X the Kasami-Welch variety as defined in section 1. The following fact, due to Janwa and Wilson [13], is known.

If $t = 2^{2k} - 2^k + 1$, then

$$\phi_t(x, y) = \prod_{\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2} P_\alpha(x, y) \quad (5)$$

where $P_\alpha(x, y)$ is absolutely irreducible of degree 2^k+1 over \mathbb{F}_{2^k} . Furthermore, P_α satisfies $P_\alpha(x, 0) = (x + \alpha)^{2^k+1}$.

Let $p = (1, 1)$, a singular point of X (Janwa and Wilson in [13, 12], and consequently Hernando and McGuire in [11], classified the singularities of X). One of our main results is that $p \in P_\alpha$, for all $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$.

Let

$$\phi_t(x, y) = \prod_{\alpha \in [\alpha]} Q_\alpha(x, y) \quad (6)$$

where $[\alpha]$ is the conjugate class of α under the action of the Frobenius automorphisms (counting in the product only one representative for each conjugacy class); i.e., $Q_\alpha \in \mathbb{F}_2[x, y]$ is the product of conjugate absolutely irreducible factors P_α . Since $Q_\alpha(p) = 0$ implies that $Q_\beta(p) = 0$ for all $\beta \in [\alpha]$,

then the fact that $p \in P_\alpha$, for all $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$, is proved if we prove that $p \in Q_\alpha$, for all $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$.

Grouping equation (6) in symmetric factors:

$$\phi_t(x, y) = \prod_{\alpha \in [\alpha]} Q'_\alpha(x, y) \quad (7)$$

where $Q'_\alpha = Q_\alpha$ if Q_α is symmetric and $Q'_\alpha = Q_\alpha \overline{Q_\alpha}$ if Q_α is not symmetric (the overline means its symmetric pair). Thus it is enough to prove that $p \in Q'_\alpha(x, y)$, since $Q_\alpha(p) = 0$ implies $\overline{Q_\alpha}(p) = 0$. Because the aim is to prove that $p \in P_\alpha$, for all $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$, that is $p \in Q_\alpha$ for all $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$ (as Q_α is a relabeling of the set P_α .)

Lemma 1. $Q'_\alpha(x, y)$ has an odd number of terms of the form $x^m y^m$ for $m \geq 1$. We will refer them as "equal degree" (ED) terms.

Proof. Since $\phi_t(0, 0) = 1$ and $\phi_t(x, y)$ is symmetric, by manipulation in (2),

$$\phi_t(x, y) = F_{2^{2k}-2^k-2}(x, y) + \cdots + F_i + \cdots + F_1(x, y) + 1, \quad (8)$$

where $F_i(x, y)$ is 0 or a symmetric homogeneous polynomial of degree i .

By the symmetry of ϕ_t , the number of non-ED terms of the form x^m, y^n or $x^n y^m$ (where $m, n > 0$), will occur in pairs in $\phi_t(x, y)$. Since $p \in X$, the multiplicity of p , m_p , is a positive number. Then, $\phi_t(x+1, y+1) = G_{2^{2k}-2^k-2}(x, y) + \cdots + G_i(x, y) + \cdots + G_{m_p}(x, y)$, where $G_i(x, y)$ is 0 or a symmetric homogeneous polynomial of degree i .

CLAIM: $\phi_t(x, y)$ should have an odd number of ED terms.

To prove this, we first observe that since each term (ED or non-ED) of $\phi_t(x, y)$ produce a constant term 1 in the expansion of $\phi_t(x+1, y+1)$, because for $m, n > 0$, $(x+1)^m, (y+1)^m, (x+1)^m(y+1)^n$, and $(x+1)^n(y+1)^m$, all equal 1, when $(x, y) = (0, 0)$. Now in the term by term expansion of $\phi_t(x+1, y+1)$, the non-constant non-ED terms, since they occur in pairs, contribute 0 (mod 2), to the constant term. Therefore, the ED terms would have to contribute 1 (mod 2) in this expansion of $\phi(x+1, y+1)$ for the constant term to vanish. Therefore $\phi_t(x, y)$ must have an odd number of duplicate ED terms.

Proof of the Lemma: Now, since $\phi_t(x, y)$ has an odd number of ED-terms, and since it is the product of Q'_α , we show that these facts force each Q'_α to have an odd number of ED-terms. Suppose that $\phi_t(x, y)$ were a

product of two distinct symmetric polynomials A_α and A_β , with the number of ED-terms t_α and t_β , then in the product $A_\alpha(x, y)A_\beta(x, y)$, we would get $t_\alpha t_\beta$ ED-terms. Since A_α and A_β are symmetric, and since each non-constant non-ED term in each A_α and A_β occur in pairs, we get a contribution of an even number of ED-terms, say $2t_{\alpha\beta}$. Since $\phi_t(0, 0) = 1$, we have $A_\alpha(0, 0) = A_\beta(0, 0) = 1$, and in the product, the constant terms contribute $t_\alpha + t_\beta$ number of ED terms. Therefore, the number of ED-terms in the product is $t_\alpha t_\beta + t_\alpha + t_\beta + 2 * t_{\alpha\beta}$, and this number is odd if and only if each t_α and t_β is odd. We have shown that the property of odd number of ED-terms propagate to its two distinct factors that are symmetric and have constant term 1. Since each $Q'_\alpha(x, y)$ is symmetric and has constant term 1, we can group them in two distinct factors A_α and A_β , and we conclude by induction that each $Q'(x, y)$ has an odd number of Ed-terms. \square

Two or more curves f_i are said to intersect transversally at a point p if p is a simple point of each f_i and if the tangent lines to f_i at p are pairwise distinct.

The next theorem is our main result.

Theorem 4. *In the absolutely irreducible factorization of Equation (5), the components $P_\alpha(x, y)$, $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$, intercept transversally at $p = (1, 1)$.*

Proof. By (3), the constant term of ϕ_t is 1. Then, as a factor of ϕ_t over $\mathbb{F}_2[x, y]$, the constant term of $Q'_\alpha(x, y)$ is also 1, for all $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$. By Lemma 1, $Q'_\alpha(x, y)$ has an odd number of ED terms. Then, by direct computation, for all $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$, the constant term of $Q'_\alpha(x + 1, y + 1)$ is zero. Thus, for all $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$, $m_p(Q'_\alpha(x, y)) \geq 1$.

Therefore, for all $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$, $p \in Q'_\alpha(x, y)$, and if $Q'_\alpha = Q_\alpha \overline{Q_\alpha}$, then $p \in Q_\alpha$ and $p \in \overline{Q_\alpha}$. Therefore, $p \in Q_\alpha$ for all suitable α . Therefore, we conclude that $p \in P_\alpha(x, y)$, as required.

In addition, as shown by Janwa, McGuire and Wilson in [12], $m_p(\phi_t) = 2^k - 2$ and the "tangent cone" factors into $2^k - 2$ different linear factors. We conclude that, the components P_α intersect transversally at p . \square

4 Two new families of absolutely irreducible Kasami-Welch degree polynomials

As some applications of Theorem 4 we have:

Theorem 5. *Let the Kasami-Welch degree polynomial $f(x) = x^{2^{2k}-2^k+1} + h(x) \in L[x]$, where $d = \deg(h) \equiv 3 \pmod{4}$. Then $\phi(x, y)$ is absolutely irreducible.*

Proof. Supposing, by the way of contradiction (as in [4, 5, 6]), that $\phi(x, y)$ factor as $\phi(x, y) = P(x, y)Q(x, y)$, where P and Q are non constant polynomials. Then, because of the factorization of $\phi_{2^{2k}-2^k+1}$ as a product of absolutely irreducible different factors, the following system is obtained (see the previous references):

$$P_s Q_t = \prod_{\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2} P_\alpha(x, y), \quad (9)$$

$$P_s Q_{t-e} + P_{s-e} Q_t = a_d \phi_d(x, y) \quad (10)$$

where $P = P_s + P_{s-1} + \dots + P_0$, $Q = Q_t + Q_{t-1} + \dots + Q_0$, $2^{2k} - 2^k + 1 > s \geq t > 0$ and $e = 2^{2k} - 2^k + 1 - d$.

By Theorem 4, $p = (1, 1) \in P_\alpha(x, y)$ for all $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$. Then, by the absolute irreducible factorization in (9), $p \in P_s, p \in Q_t$. Then by Equation (10), $p \in \phi_d(x, y)$. Which is a contradiction, since p does not belong to $\phi_d(x, y)$ as demonstrated by Janwa and Wilson in [13]. \square

For the case $d \equiv 1 \pmod{4}$, it happens that $p \in \phi_d$ and $m_p(\phi_d) = 2^i - 2$, where $d = 2^i l + 1$ (l an odd number), as shown by Hernando and McGuire in [11]. The following is our result for this case.

Theorem 6. *Let the Kasami-Welch degree polynomial $f(x) = x^{2^{2k}-2^k+1} + h(x) \in L[x]$, where $d = \deg(h) \equiv 5 \pmod{8}$. If $d < 2^{2k} - 3(2^k) - 1$ and $(\phi_{2^{2k}-2^k+1}, \phi_d) = 1$, then $\phi(x, y)$ is absolutely irreducible.*

Proof. Supposing as in Theorem 5 that $\phi(x, y)$ factors as $P(x, y)Q(x, y)$ and using the same arguments used there, we get the system:

$$P_s Q_t = \prod_{\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2} P_\alpha(x, y), \quad (11)$$

$$P_s Q_{t-e} + P_{s-e} Q_t = a_d \phi_d(x, y) \quad (12)$$

where s, t, e are in the previous theorem. Let $p = (1, 1)$ and let us consider the following two cases to prove the theorem.

Let $t > 2(2^k + 1)$. Then using Theorem 4 and the fact that P_α are absolutely irreducible polynomials of degree $2^k + 1$, from (10) we have that $m_p(Q_t) > 2$, $m_p(P_s) > 2$. This implies also that $m_p(P_s Q_{t-e} + P_{s-e} Q_t) = m_p(\phi_d(x, y)) > 2$. Contradicting that $m_p(\phi_d) = 2$ (for $d \equiv 5 \pmod{8}$), $m_p(\phi_d) = 2^2 - 2$).

On the other hand, let $t \leq 2(2^k + 1)$. Since $d < 2^{2k} - 3(2^k) - 1$, then $e > 2^{2k} - 2^k + 1 - (2^{2k} - 3(2^k) - 1) > 2^{k+1} + 2$ and $t < e$. The Equation (12) becomes $P_{s-e} Q_t = a_d \phi_d(x, y)$, which contradicts the relatively prime hypothesis.

□

5 Towards the conjecture of exceptional APN functions

As an application of Theorems 5 and 6, the following two theorems contribute substantially on the hardly studied case of the APN function conjecture, the Kasami-Welch case.

Theorem 7. *Let the Kasami-Welch degree polynomial $f(x) = x^{2^{2k}-2^k+1} + h(x) \in L[x]$ where $\deg(h) \equiv 3 \pmod{4}$. Then f is not exceptional APN.*

Theorem 8. *Let the Kasami-Welch degree polynomial $f(x) = x^{2^{2k}-2^k+1} + h(x) \in L[x]$ where $\deg(h) \equiv 5 \pmod{8}$, $d < 2^{2k}-3(2^k)-1$. If $(\phi_{2^{2k}-2^k+1}, \phi_d) = 1$, then f is not exceptional APN.*

Some remarks

In Theorem 8, one of the conditions for $f(x)$ not to be exceptional APN is that $(\phi_{2^{2k}-2^k+1}, \phi_d) = 1$. There are many cases for this to happen, for example, when ϕ_d is absolutely irreducible (this follows from the absolutely irreducible factorization of $\phi_{2^{2k}-2^k+1}(x, y)$ as given in Equation (5)). Although it is well known that for $d \equiv 5 \pmod{8}$, ϕ_d is not always absolutely irreducible, there are many cases where it is. In [12], Janwa and Wilson proved, using different methods including Hensel's lemma implemented on a computer, that $\phi_d(x, y)$ is absolutely irreducible for all $3 < d < 100$, provided that d is not a Gold or a Kasami-Welch number. They also showed that for

infinitely many values of d , ϕ_d is non-singular, and therefore absolutely irreducible. Recently Férard, in [9], established sufficient conditions for $\phi_d(x, y)$ to be absolutely irreducible, when $d \equiv 5 \pmod{8}$. Among Férard's results, by the aid of SAGE, he showed that for these type of numbers, $\phi_d(x, y)$ is absolutely irreducible for all d , $13 < d < 208$. In [9] an infinite family of integers d is given so that ϕ_d is not absolutely irreducible. (We note that Hernando and McGuire, in [11], has first showed, by using MAGMA, that $\phi_{205}(x, y)$ is not irreducible).

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